RUNGE - KUTTA METHODS

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INTRODUCTION

- In numerical analysis, the **Runge–Kutta methods** are a family of iterative methods used for obtaining the approximate solutions of ordinary differential equations (ODE).
- These methods were developed around 1900 by the German mathematicians C. Runge and M. W. Kutta.

FIRST ORDER METHOD (EULER METHOD)

- From any point on a curve, you can find an approximation of a nearby point on the curve by moving a short distance along a line tangent to the curve.
- Consider the initial value problem (IVP) of the form:

$$y'(t) = f(t, y(t)), \qquad y(t_0) = y_0,$$

- Replace the derivative $\boldsymbol{y}'$ by the finite difference approximation:

$$y'(t) \approx \frac{y(t+h) - y(t)}{h}$$

- Rearranging and substituting from IVP:

$$y(t+h) \approx y(t) + hf(t, y(t))$$

- Choosing a step size ***h*** construct the sequence t_0, $t_1 = t_0 + h$, $t_2 = t_0 + 2h$, … and denoting by y_n a numerical estimate of the exact solution $y(t_n)$ the recursive formula follows:

$$y_{n+1} = y_n + hf(t_n, y_n)$$

- This is the *Euler method* (or *forward Euler method)* and is named after Leonhard Euler who described it in 1768.

SECOND ORDER METHOD (HEUN'S METHOD)

The second order Runge – Kutta method approximates the solution of the ODE as :

$$y_{i+1} = y_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + h, y_i + k_1 h)$$

Figure 1 in next slide diagrammatically shows the numerical approximation.

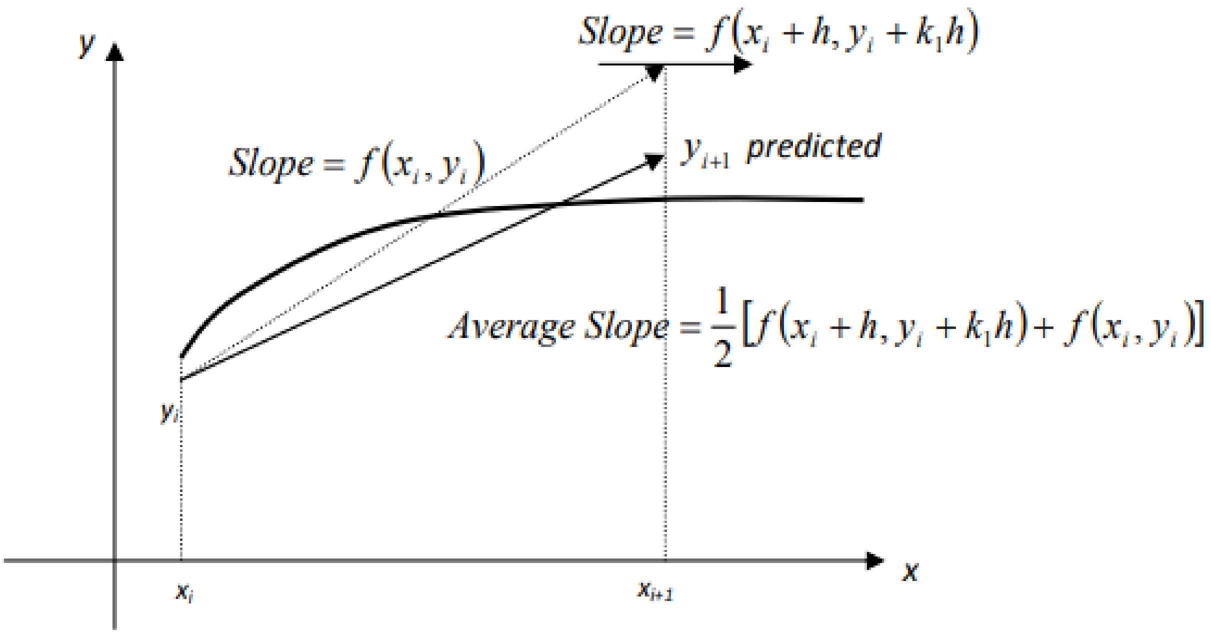

Figure 1 Runge-Kutta 2nd order method (Heun's method).

THIRD ORDER METHOD (KUTTA'S METHOD)

The third order Runge – Kutta method approximates the solution of the ODE as :

$$y_{n+1} = y_n + \frac{h}{6}[k_1 + 4k_2 + k_3],$$

where

$$k_1 = f(x_n, y_n),$$

$$k_2 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1),$$

$$k_3 = f(x_n + h, y_n - hk_1 + 2hk_2).$$

FOURTH ORDER METHOD

The fourth order Runge – Kutta method approximates the solution of the ODE as :

$$y_{n+1} = y_n + \tfrac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$
$$t_{n+1} = t_n + h$$

where

$$k_1 = h\,f(t_n, y_n),$$
$$k_2 = h\,f\left(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right),$$
$$k_3 = h\,f\left(t_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right),$$
$$k_4 = h\,f\left(t_n + h, y_n + k_3\right).$$

Problem:

Find the approximate solution of the initial value problem

$$\frac{dx}{dt} = 1 + \frac{x}{t}, \quad 1 \leq t \leq 3$$

with the initial condition

$$x(1) = 1,$$

using the Runge-Kutta fourth order with step size of $h = 1$.

Solution: The fourth order formula is

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

where

$$\begin{aligned} k_1 &= hf(t_i, x_i), \\ k_2 &= hf\left(t_i + \frac{h}{2}, x_i + \frac{k_1}{2}\right), \\ k_3 &= hf\left(t_i + \frac{h}{2}, x_i + \frac{k_2}{2}\right), \\ k_4 &= hf(t_i + h, x_i + k_3). \end{aligned}$$

Substituting for k_1, k_2, k_3, k_4:

$$\begin{aligned} k_1 &= hf(t_0, x_0) = f(1,1) = 2, \\ k_2 &= hf\left(t_0 + \frac{h}{2}, x_0 + \frac{k_1}{2}\right) = f(1.5, 2) = 2.333333, \\ k_3 &= hf\left(t_0 + \frac{h}{2}, x_0 + \frac{k_2}{2}\right) = 2.444444, \\ k_4 &= hf(t_0 + h, x_0 + k_3) = 2.722222. \end{aligned}$$

Computing y_1:

$$\begin{aligned} y_1 &= 1 + \frac{1}{6}(2 + 2 * 2.333333 \\ &+ 2 * 2.444444 + 2.722222) \\ &= 3.379630. \end{aligned}$$

Now, further computations of y_2, y_3, … can be done by iterating the previous update procedure.

BUTCHER TABLEAU

A Butcher table of the form

$$\begin{array}{c|cccc} c_1 & a_{1,1} & a_{1,2} & \cdots & a_{1,s} \\ c_2 & a_{2,1} & a_{2,2} & \cdots & a_{2,s} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_s & a_{s,1} & a_{s,2} & \cdots & a_{s,s} \\ \hline & b_1 & b_2 & \cdots & b_s \end{array}$$

is a simple mnemonic device for specifying a Runge – Kutta method

$$y_{n+1} = y_n + \sum_{i=1}^{s} b_i k_i$$

where for $1 \le i \le s$,

$$k_i = hf\left(x_n + c_i h, y_n + \sum_{j=1}^{s} a_{i,j} k_j\right).$$

BUTCHER TABLEAU FOR THE PREVIOUSLY DISCUSSED METHODS

- FIRST ORDER (EULER METHOD)

$$\begin{array}{c|c} 0 & 0 \\ \hline & 1 \end{array}$$

- SECOND ORDER (HEUN'S METHOD)

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & 1/2 & 1/2 \end{array}$$

- **THIRD ORDER (KUTTA'S METHOD)**

$$
\begin{array}{c|ccc}
0 & 0 & 0 & 0 \\
1/2 & 1/2 & 0 & 0 \\
1 & -1 & 2 & 0 \\
\hline
 & 1/6 & 2/3 & 1/6
\end{array}
$$

- **FOUTH ORDER**

$$
\begin{array}{c|cccc}
0 & 0 & 0 & 0 & 0 \\
1/2 & 1/2 & 0 & 0 & 0 \\
1/2 & 0 & 1/2 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 \\
\hline
 & 1/6 & 1/3 & 1/3 & 1/6
\end{array}
$$

REFERENCES

- https://en.wikipedia.org/wiki/Runge%E2%80%93Kutta_methods
- https://en.wikipedia.org/wiki/Numerical_methods_for_ordinary_differential_equations
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- https://wiki.math.ntnu.no/_media/tma4125/2017v/butcher.pdf
- https://www.youtube.com/watch?v=hGN54bkE8Ac
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Reference for stability of Runge – Kutta methods:

- https://en.wikiversity.org/wiki/Numerical_Analysis/stability_of_RK_methods#Example:finding_the_stability_polynomial_for_RK4's_methods
- https://youtu.be/cigFwhrQa3E

THANK YOU